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Multi-agent–based Order Book Model of financial markets

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Abstract. – We introduce a simple model for simulating financial markets, based on an order book, in which several agents trade one asset at a virtual exchange continuously. For a stationary market the structure of the model, the order flow rates of the different kinds of order types and the used price time priority matching algorithm produce only a diffusive price behavior. We show that a market trend, *i.e.* an asymmetric order flow of any type, leads to a non-trivial Hurst exponent for the price development, but not to "fat-tailed" return distributions. When one additionally couples the order entry depth to the prevailing trend, also the stylized empirical fact of "fat tails" can be reproduced by our Order Book Model.

Introduction. – The price development of assets on financial markets are determined by the superposition of the actions of market participants creating offer and demand for a financial asset similar to the emergence of macroscopic properties from microscopic interaction in statistical physics.

A few decades ago Mandelbrot [1, 2] was already able to show that financial price movements exhibit not a Gaussian behavior as assumed in economy but a more complex behavior. His work was based on datasets of limited length. In accordance with the technological improvements in computing resources, trading processes were adapted in order to create full electronic market places. Thus, the available amount of historic price data increased impressively. Merely one century ago traders only had the possibility to rely on daily datasets, whereas today records in beats of milliseconds can be analyzed. As a consequence the results achieved by Mandelbrot were confirmed and the question arises whether the established concepts in economy are stress stable or not.

If one investigates real market data from a statistical point of view one can find some non-trivial properties. Such market- and time-independent common properties are called stylized empirical facts [3–6]. This group of properties contains the important elements of significant autocorrelations for very small intra day time scales and a "fat-tailed" distribution of asset returns, *i.e.*, the probability of extreme price movements is larger than in the normal distribution. Another stylized fact is the positive autocorrelation of volatility over significant time

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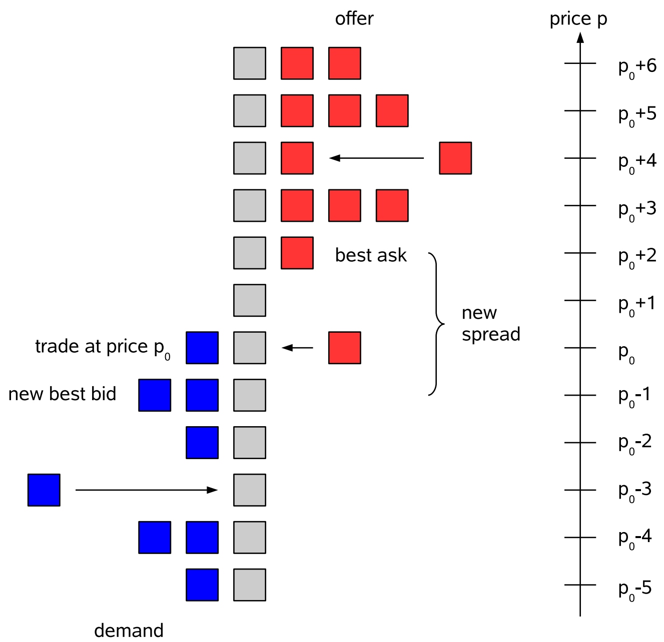

Fig. 1 – Structure of the order book: At any discrete price level limit orders will be appended chronologically in the queue of that price tick which corresponds to the price of the limit order. Thus, a price time priority matching algorithm is realized. The two orders at price p_0, on the demand side and offer side, respectively, will be matched against each other and so a trade at price p_0 is established. The spread increases after the trade to three ticks.

scales, which is known as volatility clustering. In summary, one observes an anti-persistent price behavior on short time scales, a persistence for medium time scales and a diffusive process for long time scales.

An early contribution of physicists to the modeling of financial markets using multi-agent models can be found in an interesting publication by Bak *et al.* [7]. The Bak model is based on the reaction diffusion process $A + B \rightarrow 0$ and is able to reproduce some stylized facts, *e.g.* a non-trivial Hurst exponent $H \neq 1/2$ and "fat tails". Maslov [8,9] later published an alternative model in which he introduces two kinds of orders, limit orders and market orders. A more sophisticated model was proposed by Challet and Stinchcombe [10], who added more realistic rules to the model, *e.g.* a Poisson order cancellation process. Several other agent-based market models published in the last years [11–15] were always aimed at reproducing some of the empirical stylized facts. Recently, it has become apparent that a successful modeling of financial markets has to focus in more detail on the order book structure and the order matching mechanisms in operation at financial exchanges.

Definition of the Order Book Model. – At an electronic financial exchange, a central order book stores offers and demands of the various traders and enables a continuous trading which is called continuous double auction. Prices are given as multiples of a minimum price change unit, the tick size. The highest demand price is called the best bid p_{b} and the lowest offer the best ask p_{a}, respectively. The non-zero gap between these is the spread, as shown in fig. 1. Our aim is to create an Order Book Model exactly resembling the order book at a real exchange. In our model, we only use one asset. Like in real markets, we differentiate two kinds of order

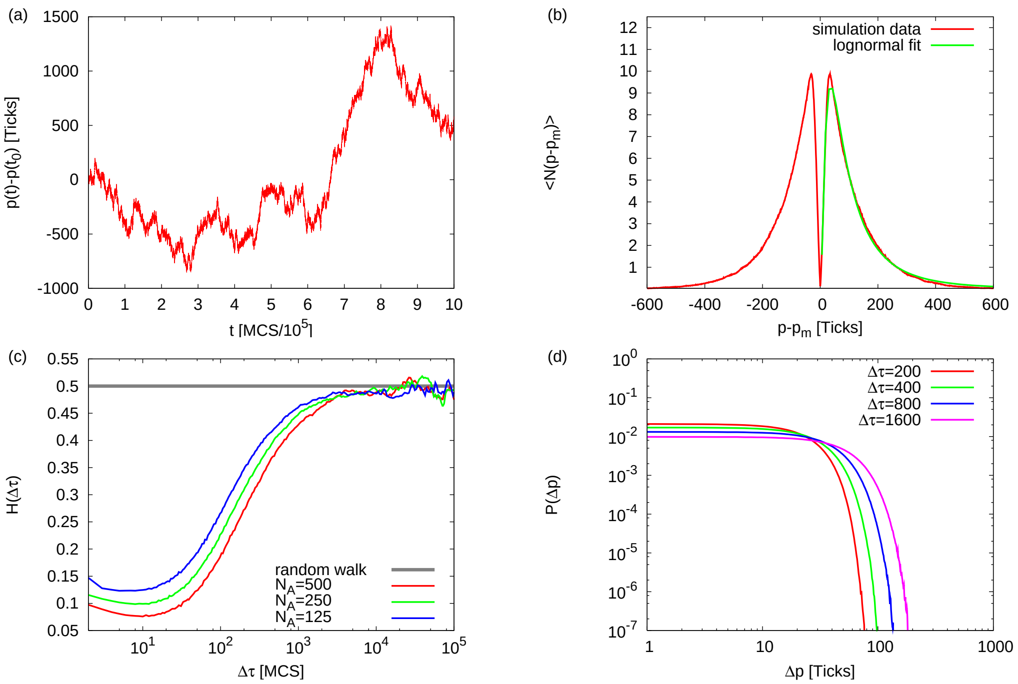

Fig. 2 – (a) For $\alpha = 0.15$, $\mu = 0.025$, $\delta = 0.025$, $\lambda_0 = 100$, $N_{\mathrm{A}} = 250$, and $q_{\text{provider}} = q_{\text{taker}} = 0.5$, one can see the price development for an exemplary sequence of 10^6 MCS. (b) Corresponding order book depth averaged over 10^4 MCS for $N_{\mathrm{A}} = 500$. (c) The Hurst exponent $H\,(\Delta\tau)$ is shown for different numbers of agents N_{A}. On long time scales, the process reaches a diffusive regime. For comparison, also the Hurst exponent of the random walk is presented. (d) Distributions of returns for different values of $\Delta\tau$ and for $N_{\mathrm{A}} = 500$.

types in our model. On the one hand, market participants can enter limit orders. Limit orders are executed at the assigned limit or some better price. On the other hand, market orders have no limit price and these orders are matched immediately at the best available market price, *i.e.*, at best ask and best bid, respectively. According to these two types of orders, we differentiate the agents on the basis of their allowed order types. In order to earn the spread, N_A liquidity providers only submit limit orders around the midpoint $p_{\mathrm{m}} = (p_{\mathrm{a}} + p_{\mathrm{b}})\,/2$ at a rate α with a probability of q_{provider} to enter on the bid side and $1 - q_{\text{provider}}$ on the ask side, respectively. N_{A} liquidity takers perform market orders with the market order rate μ. The market order of the liquidity taker is a buy order with probability q_{taker} and a sell order with $1 - q_{\text{taker}}$. In a first approach, we simply set $q_{\text{provider}} = q_{\text{taker}} = 1/2$. In addition, limit orders may be deleted by expiring or being canceled. These orders are removed with a probability δ per time unit. The order volume is restricted to one asset unit. Our matching algorithm for the orders provides a price time priority which is usually found for most assets in real markets(^{1}).

Simulation results. – We first assumed an unrealistic IID (independent identically distributed) limit order flow over an interval with a length of order $2p_{\text{int}}$, *i.e.*, liquidity providers enter their buy orders in the interval $[p_{\mathrm{a}} - 1 - p_{\text{int}}, p_{\mathrm{a}} - 1]$ on each price level with the same

(^{1})There also exist other matching algorithms in real electronic markets, *e.g.*, the pro-rata matching algorithm that only contains a price priority. More information can be found on the website of Eurex (European Derivatives Exchange) `www.eurexchange.com`.

probability and sell orders in the interval $[p_\mathrm{b}+1, p_\mathrm{b}+1+p_\mathrm{int}]$, respectively. For this case we were able to reproduce the results of [16, 17] where an equivalent microscopic dynamical statistical model for the continuous double auction under the assumption of IID random order flow is investigated for $p_\mathrm{int} \to \infty$.

However, a more realistic assumption for the order entry depth is an exponentially distributed depth structure with parameter $\lambda(t) = \lambda_0$ to adapt the model to real market behavior. For different N_A, this agent-based realization of a central order book leads to results shown in fig. 2. In our simulations, the order book depth reaches an equilibrium state after some thousand Monte Carlo steps (MCS) which can be phenomenologically described by a lognormal distribution as shown in fig. 2b. Each simulation is averaged over 50 runs. Each run consists of 10^6 MCS.

The Hurst exponent H is defined by the relationship $\langle(\Delta p)^2\rangle^{1/2}(\Delta\tau) \propto \Delta\tau^H$. In fig. 2c $H(\Delta\tau)$ increases to $H = 1/2$ and remains there for medium and long time scales. For short time scales, one obtains an anti-persistent behavior. Distributions of price increments $P(t+\Delta\tau) - P(t)$ are identical to return distributions, because the simplification of omitting logarithmic expressions is correct for short times and periods without crash events [18]. Since our price movements are independent of the start price, this assumption is satisfied.

The total number of orders $N(t)$ in the order book at some time t is important for the simulation stability. If the stationary order book depth is too small, a fluctuation in price movement could empty either the bid or the ask side of the book leading to a crash or a price explosion. $N(t+1)$ can be characterized recursively by the different order rates α, δ, μ and the number of agents N_A:

$$N(t+1) = (N(t) + \alpha N_\mathrm{A}) - (N(t) + \alpha N_\mathrm{A})\,\delta - \mu N_\mathrm{A}. \tag{1}$$

In equilibrium, which is reached after some thousand MCS in our simulations, one gets:

$$\frac{N_\mathrm{eq}}{N_\mathrm{A}} = \alpha\left(\frac{1}{\delta} - 1\right) - \frac{\mu}{\delta}. \tag{2}$$

If defining an effective limit order rate $\alpha^* = \alpha(1-\delta)$, one obtains:

$$\frac{N_\mathrm{eq}}{N_\mathrm{A}}\,\delta = \alpha^* - \mu. \tag{3}$$

It is clear that $\alpha^* > \mu$ and $\delta > 0$ must hold to reach a stable order book. This consideration and an initial investigation of the parameter space lead to our choice of the parameter set in fig. 2 as a representative set. To ensure stability of the simulation, the order book has to be filled to a sufficient depth before one allows the first market order to be placed. In our Order Book Model this problem is solved by a pre-opening sequence of 10 MCS, in which only limit orders can be submitted around the given start price $p(t_0)$ before the actual simulation starts. We use an order book with 2×10^6 price ticks with $p(t_0) = 1 \times 10^6$.

Asymmetric perturbations in our symmetric model. – In the sections above, we used $q_\mathrm{provider} = q_\mathrm{taker} = 1/2$, *i.e.*, a new order enters on the bid side and on the ask side with equal probability. In this balanced case, the Order Book Model shows a very symmetric construction. In an extension, we replace this constant side probability by two different approaches with asymmetric perturbations: In a first approach, a stochastic perturbation is added to our Order Book Model. The simplest case is to create a random walk of the variable q_taker, starting at $q_\mathrm{taker}(t = t_0) = 1/2$ and altering q_taker with a step size $\pm\Delta s$ in each time step. In order to avoid a collapse of the order book we use reflecting boundary conditions with

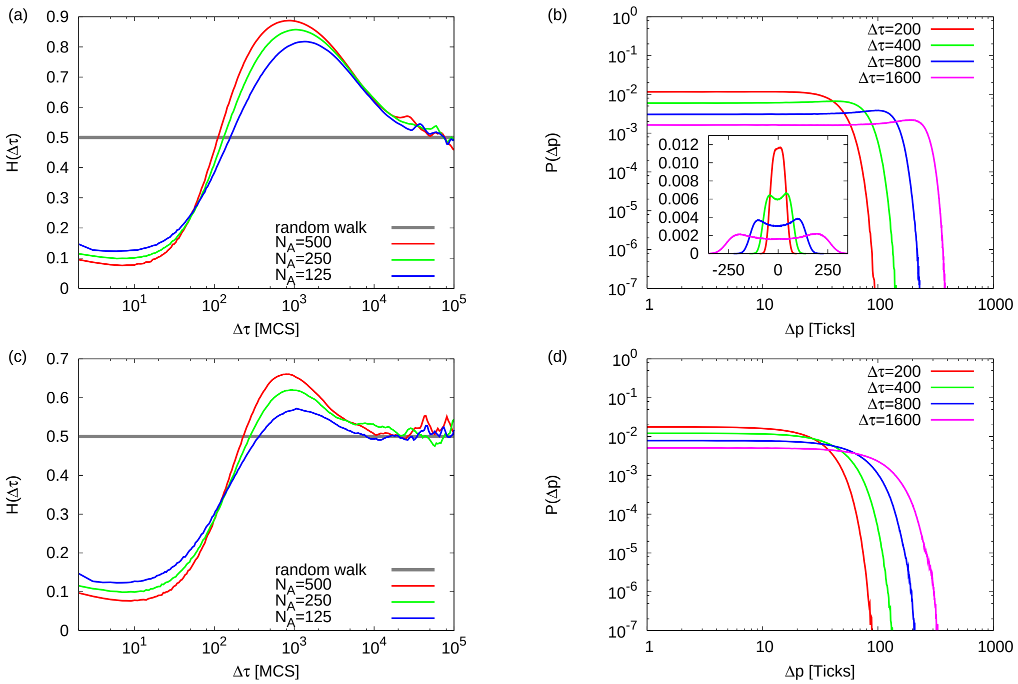

Fig. 3 – Stochastic perturbation: (a) *Ceteris paribus* $q_{\mathrm{taker}}(t)$ is modeled by a bounded random walk. $H(\Delta\tau)$ is shown for different N_{A}. After an extreme superdiffusive price development for medium time scales, the process converges against a random walk for long time scales. (b) Bimodal return distributions are obtained for the bounded random walk with $N_{\mathrm{A}} = 500$. (c) More realistic behavior of the Hurst exponent on medium time scales for modeling $q_{\mathrm{taker}}(t)$ by a feedback random walk. (d) Return distributions do not show “fat tails”. Approximately a Gaussian behavior can be observed for the feedback random walk with $N_{\mathrm{A}} = 500$.

$q_{\mathrm{taker}} \in [1/2 - S, 1/2 + S]$. This approach results in a non-trivial Hurst exponent, as shown in fig. 3a for $\Delta s = 0.001$ and $S = 0.05$. For medium time scales, we find a Hurst exponent increasing up to 0.9. For long time scales, we get a diffusive process with $H = 1/2$. Although one expects a superdiffusive process for medium time scales in real markets, this modification is not an acceptable possibility for modeling real market behavior. As known from [1,2] and from [19] for foreign exchange currency, a maximum of $H_{\max} \approx 0.6$ can be expected. As shown in a recent paper [20], $H > 1/2$ does not necessarily imply long-time correlations, but simply means that the underlying Markov process has non-stationary increments, as is the case in this change of our model. A more critical point is the shape of the distribution of the price increments, as shown in fig. 3b. For this realization of the asymmetry, we obtain a bimodal return distribution, which has nothing in common with the distributions found in financial markets.

Following this line of thinking, we next modeled q_{taker} by a mean reverting random walk. The probability for getting closer to the initial point of 1/2 is given by $1/2 + |q_{\mathrm{taker}}(t) - 1/2|$. This modification also leads to a non-trivial Hurst exponent for $\Delta s = 0.001$ and $S = 1/2$, as shown in fig. 3c, which now is closer to the experimental behavior. However, the distributions of returns shown in fig. 3d still are approximately Gaussian, showing that $H > 1/2$ and “fat-tailed” return distributions are two independent stylized facts. These findings are not changed if we keep $q_{\mathrm{taker}} = 1/2$ constant and instead modify q_{provider} in the way described

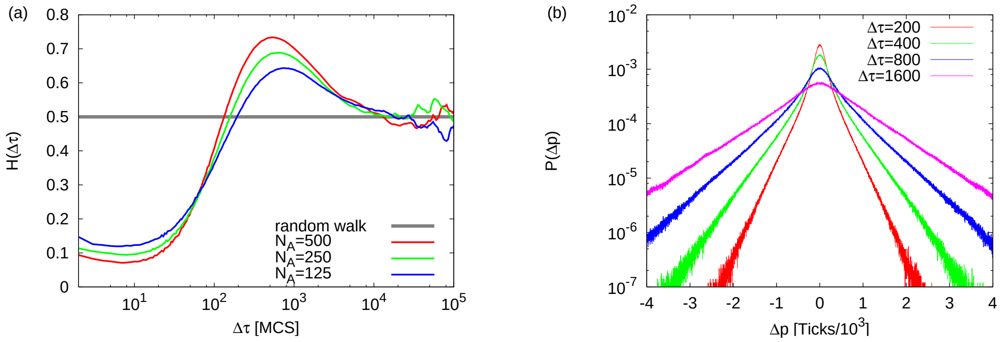

Fig. 4 – Non-static entry depth: *Ceteris paribus* a time-dependent entry depth $\lambda(t)$ is applied in combination with our modified random walk with feedback mechanism. (a) The Hurst exponent $H(\Delta\tau)$ is shown for different N_{A}, using $\lambda_0 = 100$ and $C_\lambda = 10$. Qualitatively, $H(\Delta\tau)$ shows the same behavior as before. (b) In contrast to our previous model, one can see here distinct "fat tails" in the return distributions.

above. Further feedback mechanisms on the structure of real order books have to be present to generate an increased probability for extreme fluctuations.

Non-static entry depth. – So far, a constant entry depth parameter $\lambda_0 = 100$ has been used. According to [21], one can expect a symmetry between market and limit orders from an effective costs' point of view. In general, liquidity providers do not place their limit buy and limit sell orders around the midpoint with a constant characteristic offset. In a trendless market one cannot expect large price movements and so the intention of the liquidity providers is to position the orders close to the midpoint for exploiting the small price fluctuations. In strong trend phases, however, the risk of liquidity providers increases if orders are placed close to the midpoint. Thus, liquidity providers adapt their order entry depth to decrease their market risk. So it is a reasonable assumption that λ is determined by the historic volatility. In our model, the volatility depends on the side probability, *e.g.* q_{taker}. We therefore add a time-dependent entry depth parameter $\lambda(t)$ to our modified random walk with a feedback mechanism

$$\lambda(t) = \lambda_0 \left(1 + \frac{\left|q_{\mathrm{taker}}(t) - \frac{1}{2}\right|}{\sqrt{\left\langle \left(q_{\mathrm{taker}}(t) - \frac{1}{2}\right)^2 \right\rangle}} \cdot C_\lambda\right) \tag{4}$$

in which the average $\langle \ldots \rangle$ is determined separately before the main simulation starts. For $C_\lambda = 0$, this approach corresponds to our previous approach with a static entry depth. In order to present a significant effect, we chose $C_\lambda = 10$. The results for this approach are shown in fig. 4. The time scaling of the Hurst exponent stays qualitatively the same as in our previous approach. However, the return distributions now are not Gaussian. One clearly obtains "fat tails", in accordance with results found for financial instruments in real markets. The tails can be described as straight lines in semi-logarithmic plotting, *i.e.*, the probability for large price fluctuations decreases exponentially.

Conclusion and outlook. – In this paper, we have introduced a new multi-agent–based Order Book Model, which simulates real financial markets with their underlying order book structure and with simple rules. Our model is able to reproduce important features of real markets, like the behavior of the Hurst exponent for short, medium and long time scales and "fat tails" in the return distributions.

We have shown that the occurrence of a Hurst exponent $H > 1/2$ on intermediate time scales can be traced to temporary trends leading to a non-stationary behavior of the price increments on these time scales. We have modeled these trends by a time-dependent asymmetric order flow generated by a mean reverting random walk. Furthermore, "fat tails" in return distributions can be traced to an increased entry depth of the order book generated by liquidity providers to reduce the risk of their positions in times of such a trend. Both stylized facts, which in principle are independent, are thus related by the trading strategy of "market makers". Further properties of this model will be discussed in detail in future work.

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